The Cosmological Models

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Abstract

This paper outlines my understanding of the cosmological models, mainly from a mathematical point of view. I can't count the number of times I've been frustrated by a book that discusses the cosmological models, listing the basic formulas and jumping to derived equations with no more substantiation than a clause saying "it can be shown that". This paper does that only if I can't show the process. If I *can* show how to get from here to there, then I *will*. One of the purposes of this paper is to explicitly show how those steps are accomplished. Another purpose is to combine the various bits and pieces of information from several sources into one place. Yet another is to prove the equivalence of the many forms of the equations which are found in various sources. As you read, I hope you'll check my work and let me know of anything wrong. If we can come up with an accurate and understandable version of this information, perhaps we can all refer to it five years from now when struggling with one of those "It can be shown" phrases in a new cosmology book.

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Chapter 1

Definitions

This chapter defines some notations and conventions that are used in the paper.

Billion is used to mean 10^9.

Negative exponents are often used rather than fractional representations. As a quick reminder, $a^{-b} = \frac{1}{a^b}$.

Universe is used to represent the real Universe we live in.

universe is used to mean "a model of the Universe".

The **Cosmological Principle** states that the universe, as seen by any observer at any place or time, is homogeneous and isotropic. The equations and models discussed in this paper, unless otherwise noted, assume the cosmological principle to be true.

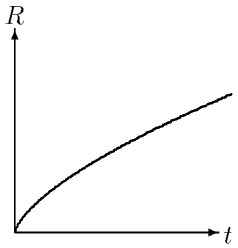
t represents time. Most of the cosmological models, especially those based on the Friedmann equation, assume a smoothly-flowing, universally-synchronized time, as advocated by Newton. This is in conflict with the ideas of Einstein, who showed that there is no universally-synchronized time. However, in this paper we are dealing with the Universe as a whole, and a universal time *does* apply in this context. Einstein's refutation of a universal time requires either an intense gravitational field or the observation of the sequence of two events from different frames of reference. The intense gravitational field is not encountered when we examine the Universe as a whole, except at very early times in a Big Bang model. I would expect that the models discussed here are not accurate at such times. No sequence of events is used to determine time in the models. Instead, the instantaneous values of various functions of time are assumed to have identical values at all points in the Universe, and the observed local values of those parameters can be used to define time.

function of time. A variable whose value changes as time passes is called a function of time. Several of the cosmological parameters are functions of time, the most obvious of which is the distance between two objects in an expanding universe, which becomes larger as time passes. These functions of time are identified as such, and employ a notation in the equations which you must understand. Let's consider a specific function of time, the Hubble Parameter, H, to learn the notation. As most of us learned in high school math, the notation $H(t)$ denotes H to be a function of t. Following a notation developed by Newton and used throughout much of physics, we will express an equation such as $\frac{1}{2}H^2(t)r^2(t) - \frac{4\pi}{3}G\rho(t)r^2(t) = k_2$ by using the notation H as an abbreviation for $H(t)$. Notice how much simpler the equation above, $\frac{1}{2}H^2r^2 - \frac{4\pi}{3}G\rho r^2 = k_2$, now becomes. We must always remember, however, that H, r, and ρ (in this example equation) are not constants, but instead are functions of time whose value changes as time passes. This notation represents the entire range of values of the Hubble Parameter as all of time passes by. The value at a specific time, for example at $t = 0$, the Big Bang, is denoted as $H(0)$. The value at time n is denoted as $H(n)$. The value of a function of time at the present time is denoted by convention with a subscripted 0, as in H_0, which represents the current value of the Hubble parameter. The parameters which are functions of time change their value as time passes, but have the same value everywhere in the universe at any given instant of time.

$\dot{\mathbf{F}}$ A single dot above a function represents the first derivative, or rate of change of the function with respect to time. Those fancy words are calculus terms for a simple concept. If F is a function of time representing the position of an object at various times, then $\dot{F}$ represents velocity (the rate of change of position). See the brief review of calculus which follows shortly for more information.

$\ddot{\mathbf{F}}$ A double dot above a function represents the second derivative, or rate of change of the rate of change of the function with respect to time. In our above example, the rate of change of velocity is acceleration. See the brief review of calculus which follows shortly for more information.

equ(2.1)(7) is the notation used to refer to an equation, which is numbered on the right edge of the page. Within the first set of parenthesis, the first number is a chapter number, and the second is a sequential number within that chapter. The number in the second set of parenthesis is a page number. This example refers to the equation which states the value used for the gravitational constant.



MODEL# 1

Einstein-de Sitter or Flat Friedmann

$k = 0$; $\lambda = 0$; $R(0) = 0 \Uparrow$; BW; Harrison 3,B; Bondi 2ii

Model Heading Line: The specific Friedmann Cosmological Models covered in chapters 4 and 5 of this paper begin with the standardized heading shown above. The graph plots the scaling factor as a function of time. Be forewarned that the models are not drawn to the same vertical or horizontal scale, but rather to the scale that best shows the peculiarities of each individual model. The first text line contains my model number, the second shows common name(s) of the model, and the third line contains:

- the value of k, the Curvature Constant
- the value of λ, the Cosmological Constant
- Initial Conditions: The initial scaling factor of the universe is shown here, followed by the initial expansion state, either $\Uparrow$ for an initially expanding universe, $\Downarrow$ for an initially contracting universe, or $\Rightarrow$ for a universe which is initially neither expanding nor contracting.
- Harrison's Life Cycle[1]: The life cycle of a model is stated as its Beginning-Middle-End state. Beginning and End can be B (bang), S (static) or W (whimper), which represents the infinitely dense state, the lack of contraction or expansion, and an infinitesimally dense state, respectively. Middle can only have a value of static, if there is a middle in the life cycle at all.
- The diagram number in Harrison (1981), page 304, table 15.12, and the diagram letter from page 305, table 15.13, lettered A through K in the same order.
- The case identifier from Bondi (1952), pages 80-86.

Brief Review of Calculus: Before we begin, we shall review the calculus required to gain the fullest understanding of this paper. It is assumed that the reader has complete knowledge of algebra, and no review of that branch of mathematics is given. A **function** is a set of ordered pairs of numbers (t, y) such that to each value of the first variable, t, there corresponds a unique value of the second variable, y. A function, in this paper, is represented by

[1] See Harrison (1981) pages 303-304 where this is called a kinematic classification.

an equation. The functions we will be dealing with in this paper are all functions of time, therefore we will use t as the first variable in our calculus review, rather than the standard first variable, x. The **derivative** of a function is an equation which evaluates to the rate of change of the original function. If the derivative of a function is plotted on a graph, the value at each evaluation of t is equal to the slope of the original function at that same value of t. The derivative of the function $y = f(t)$ can be symbolically represented in several equivalent ways:

$$\dot{f} = \frac{d}{dt}(f) = \frac{d}{dt}(f(t)) = \frac{d}{dt}(y) = \frac{dy}{dt} = y'$$

The preferred notation used in this paper will be $\dot{f} = \frac{d}{dt}(f)$. The symbol $\frac{d}{dt}(\cdots)$ is read as *the derivative with respect to time*, and can be thought of as an operator, where the operation is performed on the function inside the parenthesis. The process of applying this operator is called **differentiation** .

If function g is the derivative of function f, then function f is the **integral** of function g. It is simply the function whose rate of change is the function being integrated. The integral of the function $y = f(t)$ can be symbolically represented in this way:

$$\int y \; dt = \int f(t) \; dt.$$

The operator here is $\int \cdots dt$, which is read as *the integral with respect to time*, and the process of applying the operator is called **integration**.

It is a legal algebraic manipulation to differentiate or integrate both sides of an equation.

Below is a list of the differentiation (and integration) rules we will use in this paper, in which the following symbols are used:

a and b are constants
e is the exponential constant
t represents the independant variable of a function
u and v are functions of t

$$\frac{d}{dt}(a) = 0 \tag{1.1}$$
$$\frac{d}{dt}(t^a) = at^{a-1} \tag{1.2}$$
$$\frac{d}{dt}(au) = a\frac{d}{dt}(u) \tag{1.3}$$
$$\frac{d}{dt}(u+v) = \frac{d}{dt}(u) + \frac{d}{dt}(v) \tag{1.4}$$
$$\frac{d}{dt}(u^a) = au^{a-1}\frac{d}{dt}(u) \tag{1.5}$$
$$\frac{d}{dt}(uv) = u\frac{d}{dt}(v) + v\frac{d}{dt}(u) \tag{1.6}$$
$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dt}(u) - u\frac{d}{dt}(v)}{v^2} \tag{1.7}$$
$$\frac{d}{dt}(log_a u) = (log_a e)\frac{1}{u}\frac{d}{dt}(u) \tag{1.8}$$

Each rule can be used in "reverse" for integration, for example, (1.2) is used in this integral:

$$\int at^{a-1}dt = t^a + b.$$

Notice that when you integrate, you must add a **constant of integration** (b in the example above) into the final result. This is because the slope of a line (the derivative) contains no information about the offset, so when you "go backwards" from a slope to the original function (when you integrate), the offset is unknown and is added in as some arbitrary constant. The constant is usually given a definite value from a known (t, y) pair. This is the reverse of equ(1.1)(5), which states that the derivative of a constant is zero.

All differentiation and integration in this paper is with respect to time.

Chapter 2

The Cosmological Parameters

This chapter contains a description of the important observable and theoretical parameters used in the cosmological models. Each parameter begins with an outdented line, of a form which explicitly states whether or not the parameter is a function of time, and the units of the parameter. An example of a function of time is $\mathbf{R}(\mathbf{t})$ and an example of a parameter which is not a function of time is $\mathbf{G}$. The square brackets which follow the parameter name indicate the units of the parameter, where M represents mass, L represents length, T represents time, and X indicates the parameter has no units (it is dimensionless).

$\mathbf{c}$ $[LT^{-1}]$ is the **speed of light**, 3×10^8 m s^{-1}.

$\mathbf{G}$ $[L^3M^{-1}T^{-2}]$ is the **gravitational constant**, which determines the strength of the gravitational force. It is assumed throughout this paper (except where explicitly noted) to be a true constant which does not change over time. In the few calculations in this paper, I assume

$$G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}. \tag{2.1}$$

$\mathbf{r}(\mathbf{t})$ $[L]$ is a **distance**.

$\mathbf{R}(\mathbf{t})$ $[X]$ is the universal **scaling factor**[1], which relates to the size of the universe and distances within the universe. The scaling factor is probably the most important parameter used in this paper, since it is the basis for almost all other parameters. To understand the scaling factor, we begin by defining the model we'll use to represent our Universe. Lets begin with an analogy. Imagine a sheet of rubber with an x-y grid and several dots drawn on it. As the rubber sheet is stretched, the coordinate of any dot does not change, therefore the **coordinate distance** between any two dots does not change, however the **true distance** between any two dots *does* change. The system we have laid out here is called a **comoving coordinate system** since the coordinate grid and the dots are moving together. The scaling factor is a function which describes how the rubber sheet is stretched and relaxed as time passes. It might be a simple linear function of time, or an exponential one, or any other function we could think of. One of our jobs in building an accurate model of the universe is to select the proper scaling function which matches the observables in the real Universe and is explainable by the known forces. The most general equation relating the scaling factor to distances is

$$\frac{r}{r_0} = \frac{R}{R_0}.$$

To simplify our mathematics, we'll lay out our coordinate system using the scale of the Universe today, we'll measure coordinate distances to galaxies now, and we'll force the scaling function to evaluate to 1 at the present time, or, in mathematical terms, force $R_0 = 1$. Now, no matter what scaling function we select, we can use the equation

$$r = r_0 R. \tag{2.2}$$

[1] See Harrison (1981), pages 217-220.

In English, this equation states that if a distance is measured as r_0 today (that is its coordinate distance), then its distance r at any other instant is simply r_0 times the value of the scaling function, R, at that other instant.

Many authors use $\mathbf{a(t)}$ to represent the scaling factor. We will use R throughout this paper.

v(t) $[LT^{-1}]$ is a **velocity**. In this paper, we ignore the peculiar velocity[2] of a particle (galaxy) and are only concerned with the component of its velocity which is due to the expansion of the universe, otherwise known as the recession velocity. This is simply the velocity at which the particle's distance from us is increasing. However, since the particle is comoving with the coordinate system, its recession velocity is simply the rate of change of R times the coordinate distance r_0, giving

$$v = r_0 \dot{R}. \tag{2.3}$$

H(t) $[T^{-1}]$ is the **Hubble Parameter** (often incorrectly called the Hubble Constant), which characterizes the rate of expansion of the universe. The Hubble Parameter is defined as

$$H \equiv \dot{R}R^{-1} \tag{2.4}$$

The current value is probably [3] $50 < H_0 < 100$ km s^{-1} Mpc^{-1}. For ease of calculation, we'll convert these values to the simpler units of s^{-1} using the conversion factor of $(10^3 \text{ m km}^{-1})/(3.1 \times 10^{22} \text{ m Mpc}^{-1})$ which converts both of the distance units into meters so the units will cancel out. This gives us a Hubble parameter in the range of $(1.6 \times 10^{-18}) < H_0 < (3.2 \times 10^{-18})$ s^{-1}.

Note that rearranging equ(2.4)(8) into the form

$$\dot{R} = RH,$$

substituting it into equ(2.3)(8)

$$v = r_0 RH,$$

and substituting equ(2.2)(7) into here gives

$$v = Hr \tag{2.5}$$

which you should recognize as the **velocity-distance law**.

Using equ(1.3)(5) for the left side and equ(1.6)(5) for the right side, we can differentiate equ(2.4)(8) to obtain

$$\dot{H} = \frac{R\ddot{R} - \dot{R}^2}{R^2},$$

and buried somewhere in this equation is an understanding of the rate of change of the Hubble Parameter; how its value changes in time.

h $[T^{-1}]$ is the **Normalized Hubble Constant**. It is often more convenient to express today's value of the Hubble Parameter (H_0) using the dimensionless constant h, which is defined with

$$H_0 = 100\ h \text{ m s}^{-1} \text{ Mpc}^{-1},$$

or

$$h = H_0/100\ h \text{ m s}^{-1} \text{ Mpc}^{-1}.$$

Assuming $50 < H_0 < 100$ km s^{-1} Mpc^{-1}, $0.5 < h < 1$ km s^{-1} Mpc^{-1}.

[2] Peculiar velocity is due to the motion of a galaxy within its cluster, and the component of the redshift due to pecular velocity is dwarfed at large distances by the component of the redshift due to the expansion of the Universe. See the definition of z (redshift).

[3] Hodge (1993) states $H_0 = 75 \pm 15\%$ km s^{-1} Mpc^{-1}. Hubble Space Telescope measurements of Cepheids in M100 (Freedman 1994) led to the conclusion that $H_0 = 82 \pm 17$ km s^{-1} Mpc^{-1}.

$\tau(\mathbf{t})$ $[T]$ is the **Hubble Period** (also known as the Hubble Time, or Expansion Time), which characterizes the time scale for the expansion of the universe at any epoch. The Hubble Period is defined as

$$\tau \equiv H^{-1} = R\dot{R}^{-1}. \tag{2.6}$$

This is *not* the age of the universe unless H is constant, but is rather an extrapolation of the current expansion rate and scaling factor of the universe backward in time. It is the age the universe would be if it had always expanded at a constant rate equal to the present rate of expansion. For expanding universes, the true age of the universe is longer than this in accelerating models, and shorter than this in decelerating models. Differentiating equ(2.6)(9) using equ(1.3)(5) for the left side and equ(1.6)(5) for the right side, gives[4]

$$\dot{\tau} = \frac{\dot{R}\dot{R} - R\ddot{R}}{\dot{R}^2} = 1 - \frac{R\ddot{R}}{\dot{R}^2} = 1 + q. \tag{2.7}$$

$\rho(\mathbf{t})$ $[ML^{-3}]$ is the average **density** of the universe. Its equation is

$$\rho = \frac{M}{\frac{4\pi}{3}r^3} = \frac{3M}{4\pi r^3} \tag{2.8}$$

where M is the mass within a spherical radius r. Since the mass of the universe remains constant as its volume increases,

$$\frac{\rho}{\rho_0} = \frac{1/R^3}{1/R_0^3}.$$

Rearranging this equation and substituting $R_0 = 1$, we arrive at

$$\rho = \rho_0 R^{-3} \tag{2.9}$$

$\rho_\mathbf{c}(\mathbf{t})$ $[ML^{-3}]$ is the **critical density** in the Einstein-de Sitter model (model# 1) which divides a closed (more dense than ρ_c) universe from an open (less dense than ρ_c) universe. Referring forward to equ(3.2)(12), which was derived without the use of ρ_c, and setting $\lambda = 0$ and $k = 0$, to force a critically-balanced (flat) universe,

$$H^2 = \frac{8\pi}{3}G\rho_c$$

which can be rearranged into

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{2.10}$$

$\mathbf{p}(\mathbf{t})$ $[MT^{-2}L^{-1}]$ is the average **pressure** of the universe. In this paper, we will be neglecting the pressure of the universe[5].

$\mathbf{q}(\mathbf{t})$ $[X]$ is the dimensionless **Deceleration Parameter**[6], which characterizes the extent to which the self-gravitation of the universe is slowing down the expansion. It is defined as

$$q \equiv -\frac{\ddot{R}}{RH^2} = -\frac{R\ddot{R}}{\dot{R}^2} = -R\ddot{R}\dot{R}^{-2}. \tag{2.11}$$

As we shall see:
- $q < 0.5$ results in an ever-expanding (hyperbolic, open) universe.
- $q = 0.5$ results in a flat (Euclidian, open) universe.
- $q > 0.5$ results in a closed (spherical) universe.

[4] See equ(2.11)(9) below for the definition of q.
[5] For a discussion of the impact of considering this parameter, see Harrison (1981), pages 327-328.
[6] See Harrison (1981), page 222 for an excellent diagram explaining the deceleration parameter.

When the rate of expansion, $\dot{R}$, is constant, then $\ddot{R} = 0$, therefore $q = 0$ (we have an open universe). If H is constant, then q is constant. When q is positive there is a deceleration of the scaling factor, and when q is negative there is an acceleration of the scaling factor. Note that in a decelerating universe, the deceleration need not be strong enough to close the universe, therefore a value of $0 < q < 0.5$ results in an ever-expanding universe for which the expansion is always decreasing but still always present. Rearranging equ(2.7)(9) we can come up with an alternate expression for q:

$$q = \dot{\tau} - 1. \tag{2.12}$$

Ω(t) $[X]$ is the **Density Parameter**, a dimensionless measure of the density of the universe which equals the ratio of the density to the critical density. We use equ(2.10)(9) to present the parameter in its most common form of

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{\rho}{\frac{3H^2}{8\pi G}} = \frac{8\pi}{3} G\rho H^{-2}. \tag{2.13}$$

Ψ(t) $[X]$ is the **Pressure Parameter**, which measures the extent to which the pressure of the matter and radiation in the universe makes a dynamic contribution to the evolution of the universe. It is defined as

$$\Psi \equiv \frac{3p}{\rho c^2}. \tag{2.14}$$

This parameter is mentioned here for completeness only, as it is never referred to again in this paper. (Read prior sentence as "I found it in a book, but I don't understand it").

k $[T^{-2}]$ is a constant of integration called the **Curvature Constant**. In Newtonian cosmology, it represents the initial energy injected into the universe. In Relativistic cosmology, it represents the curvature of space-time.

K(t) $[T^{-2}]$ is just an alternate representation of k. It is defined as

$$K \equiv kR^{-2}. \tag{2.15}$$

Note that since $R_0 = 1$, $K_0 = k$.

λ $[T^{-2}]$ is the **Cosmological Constant**[7]. This constant was added to the cosmological equations by Einstein in order to force the universe to be static, as he believed it had to be. As it turns out, it fails to accomplish that goal. The so-called lambda force enters into the cosmological equations in the same way as the gravitational force. Based on the sign of the constant, it can either reinforce (negative constant) gravity or oppose (positive constant) gravity. Based on its magnitude, it can either be a minimally-effective force or it can completely overwhelm gravity and cause runaway cosmic expansion. To obtain maximum generality, I have included the cosmological constant in my derivation of the Friedmann equation, and assume it to be zero only on the models which require it to be so. Like G, this constant is assumed in this paper to not change in time, although some non-standard cosmologies allow λ to vary over time.

$\lambda_{\mathbf{E}}$ $[T^{-2}]$ is the critical value of the Cosmological Constant which Einstein prescribed to precisely balance the pull of gravity and force a static universe. Its equation is

$$\lambda_E = \frac{k^3}{(4\pi G\rho_0)^2}, \tag{2.16}$$

which we'll derive during our discussion of model #12 as equ(5.7)(31). It can also be expressed as

$$\lambda_E = 4\pi G\rho, \tag{2.17}$$

which we'll derive during our discussion of model #12 as equ(5.8)(31). Note that even though this value exactly balances gravity, it does not force a static universe over the long term since any small density fluctuation will tend to grow and upset the critical balance.

[7]Lest we be easily persuaded by the many modern cosmology books and articles which claim we have no need nor justification for the Cosmological Constant, see Abbot 1988. The author presents some convincing arguments that the Cosmological Constant *must* be nonzero.

Chapter 3

Derivation of the Friedmann Equation

The Friedmann Equation is the central equation in Big Bang cosmology. Two independant derivations are given in this chapter. Both use **Newtonian Cosmology**, which gives identical answers to **Relativistic Cosmology**, except at very early times in a Big Bang universe[1]. The first uses no calculus and the second does. The equation can be expressed in several equivalent forms. It contains many free parameters, that is, variables which can be given any of a multitude of values, which produce a multitude of effects. One of the reasons to examine the Friedmann Equation is to adjust those free parameters to produce a model that reflects the observed characteristics of the true Universe.

Consider an arbitrary spherical shell of matter expanding into a pre-existing universe of empty space. The density in the interior of the shell is uniform and equal to the mean cosmological density. Note that this assumption is only valid if the universe is homogenous and isotropic.

Definition of terms:

M = mass interior to the shell (a constant)
m = mass of the shell (a constant)
r = radius of the shell
v = velocity of expansion of the shell
ρ = mean cosmological density
k_1 = a constant with units $[ML^2T^{-2}]$

By applying the law of energy conservation to the shell's motion, the sum of the kinetic energy of expansion $\frac{1}{2}mv^2$ and the gravitational potential energy $-\frac{GMm}{r}$ must be a constant. This is mathematically stated as[2]

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = k_1.$$

Divide both sides by m and replace mass M by density ρ times volume $\frac{4\pi}{3}r^3$ (see equ(2.8)(9)) to obtain

$$\frac{1}{2}v^2 - \frac{4\pi}{3}\frac{G\rho r^3}{r} = \frac{k_1}{m}.$$

Using the velocity-distance law, equ(2.5)(8), and cancelling out r/r gives

$$\frac{1}{2}H^2r^2 - \frac{4\pi}{3}G\rho r^2 = \frac{k_1}{m}.$$

[1] See Harrison (1981) pages 287-288, Narlikar (1977) pages 107-112, Silk (1980) pages 87-88, and numerous other sources which attest to this.

[2] We check our work by checking units$(M)(L^2T^{-2}) - \frac{(L^3M^{-1}T^{-2})(M)(M)}{L} = (ML^2T^{-2})$. Note also that if we substitude $v = c$ and $k_1 = 0$ into the equation and rearrange, we end up with $r = \frac{2GM}{c^2}$, which is the Schwarzchild Radius of a black hole.

Substituting equ(2.2)(7) into this gives

$$\frac{1}{2}H^2 r_0^2 R^2 - \frac{4\pi}{3}G\rho r_0^2 R^2 = \frac{k_1}{m}.$$

Multiplying both sides by $2r_0^{-2}R^{-2}$ and rearranging, we obtain

$$H^2 = \frac{8\pi}{3}G\rho + \frac{2k_1}{mr_0^2}R^{-2}.$$

Now lets clean-up our constant by substituting[3] $k = \frac{-2k_1}{mr_0^2}$. Note that all terms in this equation are constants, so we're just consolidating several constants into one constant. This substitution results in

$$H^2 = \frac{8\pi}{3}G\rho - kR^{-2}, \tag{3.1}$$

which is the **Friedmann Equation with Zero Cosmological Constant**. At this point, to generalize the equation, we will add in the cosmological constant in its most convenient form[4].

$$H^2 = \frac{8\pi}{3}G\rho + \frac{\lambda}{3} - kR^{-2}. \tag{3.2}$$

This is completely legal, because we can always set $\lambda = 0$ to get back to equ(3.1)(12). This equation, and the two that follow, are the most general forms of the **Friedmann Equation**. Often it is more convenient to use the following form of the Friedmann equation, which substitutes equ(2.4)(8) into equ(3.2)(12), then multiplies both sides by R^2, giving

$$\dot{R}^2 = \frac{8\pi}{3}G\rho R^2 + \frac{\lambda}{3}R^2 - k. \tag{3.3}$$

One more form of the Friedmann equation which we will use quite often substitutes equ(2.9)(9) into this equation, giving

$$\dot{R}^2 = \frac{8\pi}{3}G\rho_0 R^{-1} + \frac{\lambda}{3}R^2 - k \tag{3.4}$$

In all three versions of the Friedmann equation which we have derived above, the rate of change of the scaling factor, H^2 or $\dot{R}^2$, is equal to the sum (or difference) of three terms. Throughout this paper, we will refer to those three terms as the **gravity term**, the **lambda term**, and the **curvature term**, respectively. Since the left side of the equation is a squared value, it must always be positive. This means the sum of the three terms on the right side must always be positive. But even though both the right side (the sum of the three terms), and the left side ($\dot{R}^2$ or H^2) must be positive, you must always remember that $\dot{R}$ or H itself can be either positive (representing expansion of the universe), negative (representing contraction of the universe), or zero (representing a static universe).

Now we begin the derivation of the Friedmann equation using calculus. This derivation is completely independent from the non-calculus derivation above. We begin with the equation of motion, which states that the acceleration of a particle is equal to the gravitational field. The acceleration is given by $r_0\ddot{R}$, which is simply the distance of the particle from the center today times the acceleration of the comoving coordinate system. The gravitational field is that produced by the mass M interior to the position r is given by the expression $-\frac{GM}{r^2}$. Since the interior mass is $M = \frac{4\pi}{3}\rho r^3$ (see equ(2.8)(9)), this makes the gravitational field $-\frac{4\pi}{3}G\rho r$. Upon equating the acceleration and the gravitational field, we obtain

$$r_0\ddot{R} = -\frac{4\pi}{3}G\rho r.$$

[3] We check our work by checking units $T^{-2} = \frac{(ML^2T^{-2})}{(M)(L^2)}$.

[4] We check our work by checking units $T^{-2} = (L^3M^{-1}T^{-2})(ML^{-3}) + (T^{-2}) + (T^{-2})$.

At this point, add in the postulated lambda force, which is proportional to the distance of the particle from the center, giving

$$r_0\ddot{R} = -\frac{4\pi}{3}G\rho r + \frac{\lambda}{3}r.$$

Substitute equ(2.2)(7) twice into this equation and divide both sides by r_0 to obtain

$$\ddot{R} = -\frac{4\pi}{3}G\rho R + \frac{\lambda}{3}R. \tag{3.5}$$

Substituting equ(2.9)(9) into this equation gives

$$\ddot{R} = -\frac{4\pi}{3}G\rho_0 R^{-2} + \frac{\lambda}{3}R.$$

We need to integrate this equation to change it from an acceleration equation to a velocity equation for easier analysis. I cannot find a way to integrate it directly, however, if we multiply both sides by $\dot{R}$, I can.

$$\ddot{R}\dot{R} = -\frac{4\pi}{3}G\rho_0 R^{-2}\dot{R} + \frac{\lambda}{3}R\dot{R}.$$

Now put in the symbols to integrate the equation

$$\int \left(\ddot{R}\dot{R}\right) dt = \int \left(-\frac{4\pi}{3}G\rho_0 R^{-2}\dot{R} + \frac{\lambda}{3}R\dot{R}\right) dt.$$

Use equ(1.4)(5) to split the right side into two simpler integrals and use equ(1.3)(5) to bring the constants outside the integrals

$$\int \left(\ddot{R}\dot{R}\right) dt = -\frac{4\pi}{3}G\rho_0 \int \left(R^{-2}\dot{R}\right) dt + \frac{\lambda}{3}\int \left(R\dot{R}\right) dt.$$

Use equ(1.5)(5) three times to integrate, and add the constant of integration (any arbitrary constant we wish)

$$\frac{1}{2}\dot{R}^2 = -\frac{4\pi}{3}G\rho_0\left(-R^{-1}\right) + \frac{\lambda}{3}\left(\frac{1}{2}R^2\right) - \frac{k}{2}$$

and simplify by multiplying both sides by 2 to obtain the Friedmann equation

$$\dot{R}^2 = \frac{8\pi}{3}G\rho_0 R^{-1} + \frac{\lambda}{3}R^2 - k. \tag{3.6}$$

Which is identical to equ(3.4)(12). Since equ(3.4)(12) can be transformed into equ(3.2)(12) and equ(3.3)(12), all four of these equations are equivalent. Both the non-calculus and the calculus derivations result in the same final equations.

Having derived the Friedmann equations, we can perform some simple substitutions to express these equations in terms of the observable parameters K, H, q, and ρ. The upcoming equ(3.7)(14) and equ(3.8)(14) are the results of these manipulations. First, we start by rearranging equ(3.5)(13)

$$\frac{\lambda}{3}R = \frac{4\pi}{3}G\rho R + \ddot{R}.$$

Divide both sides by R and multiply the last term by $1 = \frac{\dot{R}^2}{\dot{R}^2} \times \frac{R}{R}$ to obtain

$$\frac{\lambda}{3} = \frac{4\pi}{3}G\rho + \frac{\ddot{R}\dot{R}^2 R}{R\dot{R}^2 R}.$$

Rearrange to

$$\frac{\lambda}{3} = \frac{4\pi}{3}G\rho - \left(\frac{\dot{R}^2}{R^2}\right)\left(-\frac{\ddot{R}R}{\dot{R}^2}\right).$$

Substituting equ(2.4)(8) and equ(2.11)(9) into this yields

$$\frac{\lambda}{3} = \frac{4\pi}{3}G\rho - H^2 q,$$

and a final multiplication by 3 gives[5]

$$\lambda = 4\pi G\rho - 3H^2 q. \tag{3.7}$$

The second equation begins by substituting equ(2.15)(10) into equ(3.3)(12) to obtain

$$\dot{R}^2 = \frac{8\pi}{3}G\rho R^2 + \frac{\lambda}{3}R^2 - KR^2.$$

Rearrange this into

$$KR^2 = \frac{8\pi}{3}G\rho R^2 + \frac{\lambda}{3}R^2 - \dot{R}^2.$$

Divide both sides by R^2 and substitute equ(2.4)(8) into here to obtain

$$K = \frac{8\pi}{3}G\rho + \frac{\lambda}{3} - H^2.$$

Substitute equ(3.7)(14) into here to get

$$K = \frac{8\pi}{3}G\rho + \frac{4\pi G\rho - 3H^2 q}{3} - H^2.$$

Finally, we can rearrange this into[6]

$$K = 4\pi G\rho - H^2(q+1). \tag{3.8}$$

[5] We check our work by checking units $T^{-2} = (L^3M^{-1}T^{-2})(ML^{-3}) - (T^{-2})$.
[6] We check our work by checking units $T^{-2} = (L^3M^{-1}T^{-2})(ML^{-3}) - (T^{-2})$.

Chapter 4

Friedmann Models with $\lambda = 0$

In the preceeding chapter, the Friedmann Equation was derived. The unknowns in equ(3.2)(12) are the free parameters λ, k, and the initial value of the scaling parameter, $R(0)$. H and ρ are parameters which must be measured. To build an accurate model of the Universe, values consistant with observations must be assigned to the free parameters. This and the next chapter use a brute-force method to narrow the possibilities by examing the universes which result from trying all combinations of the free parameters. The next chapter examines models with $\lambda \neq 0$. This chapter examines the Friedmann Equation for the models with $\lambda = 0$. First we will study the three most widely-accepted Big Bang models of the universe, then two other models with $\lambda = 0$.

Before we begin with the first model, we'll derive a few more equations **which are only valid in the $\lambda = 0$ models**. Substituting $\lambda = 0$ into equ(3.7)(14) gives

$$4\pi G\rho = 3H^2 q.$$

Substituting this into equ(3.8)(14) gives

$$K = 3H^2 q - H^2(q+1) = 3H^2 q - H^2 q - H^2 = H^2(3q - q - 1),$$

or

$$K = H^2(2q - 1). \tag{4.1}$$

This equation relates the observable quantities H_0 and q_0, providing a way to test the model. It can be used to determine the Curvature Constant, provided that today's values of the Hubble Parameter and Deceleration Parameter are known. This equation can also be derived using calculus. Recalling the Friedmann Equation (equ(3.4)(12)) with $\lambda = 0$,

$$\dot{R}^2 = \frac{8\pi}{3} G\rho_0 R^{-1} - k.$$

Multiplying both sides of the equation by R, we obtain

$$R\dot{R}^2 = \frac{8\pi}{3} G\rho_0 - kR.$$

Using equ(1.5)(5), we find the derivative of $\dot{R}^2$ is $2\dot{R}\ddot{R}$. Using this fact, and differentiating both sides of the equation using equ(1.6)(5) results in

$$(R)(2\dot{R}\ddot{R}) + (\dot{R})(\dot{R}^2) = -k\dot{R},$$

which simplifies to

$$k = -(2R\ddot{R} + \dot{R}^2).$$

In order to simplify this a bit further, we multiply one of the terms by $1 = \frac{\dot{R}^2}{\dot{R}^2}$, getting

$$k = -\left(2\frac{\dot{R}^2}{\dot{R}^2}R\ddot{R} + \dot{R}^2\right),$$

which can be rearranged to

$$k = \dot{R}^2\left(-2\frac{R\ddot{R}}{\dot{R}^2} - 1\right),$$

and substituting equ(2.11)(9) into this equation gives

$$k = \dot{R}^2(2q - 1). \tag{4.2}$$

Substituting equ(2.15)(10) and equ(2.4)(8) into this equation lets us check our work, giving

$$K = H^2(2q - 1),$$

which matches equ(4.1)(15).

The next equation we'll derive is for the density of the universe. Starting with equ(3.3)(12) with $\lambda = 0$,

$$\dot{R}^2 = \frac{8\pi}{3}G\rho R^2 - k,$$

and solving for ρ gives

$$\rho = \frac{3(\dot{R}^2 + k)}{8\pi G R^2}.$$

Substituting equ(4.2)(16), then simplifying gives

$$\rho = \frac{3(\dot{R}^2 + \dot{R}^2(2q-1))}{8\pi G R^2} = \frac{3(\dot{R}^2 + 2q\dot{R}^2 - \dot{R}^2)}{8\pi G R^2} = \frac{3q\dot{R}^2}{4\pi G R^2},$$

or

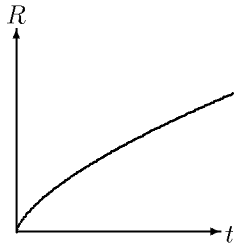
$$\rho = \frac{3qH^2}{4\pi G} \tag{4.3}$$

which is a general equation for the density at any time in a $\lambda = 0$ model.

Substituting equ(4.3)(16) and equ(2.10)(9) into equ(2.13)(10), we have

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{3qH^2/4\pi G}{3H^2/8\pi G} = 2q \tag{4.4}$$

which states that in any $\lambda = 0$ model, the density parameter is $\Omega = 2q$.



MODEL# 1

Einstein-de Sitter or Flat Friedmann

$k = 0$; $\lambda = 0$; $R(0) = 0 \Uparrow$; BW; Harrison 3,B; Bondi 2ii

We'll begin with this model for two main reasons. First of all, since both the lambda term and the curvature term in the Friedmann equation are zero, the equation becomes simpler in this model than in any other model. Secondly, as it turns out, this model is a "dividing line" between the other two widely-accepted $\lambda = 0$ models (the ones with nonzero curvature). Using equ(3.4)(12), set $k = 0$ and $\lambda = 0$ to get

$$\dot{R}^2 = \frac{8\pi}{3} G\rho_0 R^{-1}, \tag{4.5}$$

and take the square root of both sides to obtain

$$\dot{R} = \sqrt{\frac{8\pi G\rho_0}{3R}}. \tag{4.6}$$

Now we have an equation that is fairly simple and perhaps we can get a handle on what it means. One of the techniques I learned in math classes was to first try to understand an equation with all constants removed. This simplifies the equation to

$$\dot{R} \propto \sqrt{\frac{1}{R}}.$$

Reviewing some terminology, R is the scaling factor of the universe relative to its scaling factor today, and $\dot{R}$ is the rate of change of R. This equation states that the scaling factor's value determines the rate of change of the scaling factor, which "feeds back" into the scaling factor. If R is very small, lets say one trillionth, then the square root is quite large, the rate of change of R is quite large, and R becomes rapidly larger. As R approaches 1, the square root approaches 1. The rate of expansion of the universe is slowing. Once R passes 1 the square root becomes small, and the expansion slows even more. When R becomes extremely large, the square root still has an extremely small value, and the universe is still expanding, although quite slowly. Therefore, this model results in an open universe, one which ends in a whimper. Some value of R is asymptotically approached, but never quite reached.

The Einstein-de Sitter model is one of the easier models to solve for exactly (because so many of the terms are eliminated). Begin by rewriting equ(4.6)(17) as

$$\dot{R} = \sqrt{\frac{8\pi}{3} G\rho_0} R^{-1/2}.$$

Multiply both sides by $R^{1/2}$ and integrate using equ(1.5)(5) for the left side and equ(1.3)(5) for the right side, giving

$$\frac{2}{3} R^{3/2} = \sqrt{\frac{8\pi}{3} G\rho_0} t.$$

Square both sides, then multiply both sides by 9/4 to obtain[1]

$$R^3 = 6\pi G\rho_0 t^2. \tag{4.7}$$

[1] After all of this, units should be checked. $(X) = (L^3 M^{-1} T^{-2})(M L^{-3})(T^2)$.

Substituting equ(2.9)(9) and $k = 0$ into the equ(3.1)(12), the Friedmann equation with $\lambda = 0$ gives

$$H^2 = \frac{8\pi}{3} G\rho_0 R^{-3}.$$

Substituting equ(4.7)(17) into here, we obtain

$$H^2 = \frac{8\pi}{3} G\rho_0 \left(\frac{1}{6\pi G\rho_0 t^2} \right),$$

which simplifies to

$$H^2 = \frac{4}{9t^2},$$

or, after rearranging and taking the square root,

$$t = \frac{2}{3} H^{-1} = \frac{2}{3}\tau. \tag{4.8}$$

Evaluating this equation at $H_0 = 50$ km s^{-1} Mpc^{-1} $= 1.6 \times 10 - 18\mathrm{s}^{-1}$ gives

$$t = \left(\frac{2}{3}\right) \times \left(\frac{1}{1.6 \times 10^{-18}\ \mathrm{s}^{-1}} \right) = (4.17 \times 10^{17}\ \mathrm{s}) \times \left(\frac{1\ \mathrm{year}}{3.2 \times 10^{7}\mathrm{s}} \right) = 13\ \text{billion years}. \tag{4.9}$$

Evaluating this equation at $H_0 = 100$ km s^{-1} Mpc^{-1} $= 3.2 \times 10 - 18\mathrm{s}^{-1}$ gives

$$t = \left(\frac{2}{3}\right) \times \left(\frac{1}{3.2 \times 10^{-18}\ \mathrm{s}^{-1}} \right) = (2.08 \times 10^{17}\ \mathrm{s}) \times \left(\frac{1\ \mathrm{year}}{3.2 \times 10^{7}\mathrm{s}} \right) = 6.5\ \text{billion years}. \tag{4.10}$$

So, if our Universe fits the Einstein-de Sitter model, and the current Hubble Parameter is between 50 and 100, then the Big Bang happened between 6.5 and 13 billion years ago.

New let's solve for the deceleration parameter. We'll begin by assigning a temporary constant $\beta = (6\pi G\rho_0)^{1/3}$ for convenience, and substituting this constant into equ(4.7)(17)

$$R^3 = \beta^3 t^2.$$

Taking the cube root of this gives

$$R = \beta t^{2/3}. \tag{4.11}$$

Differentiating with respect to time using equ(1.3)(5) for the left side and equ(1.2)(5) for the right side gives

$$\dot{R} = \frac{2}{3} \beta t^{-1/3}. \tag{4.12}$$

Differentiating again using the same rules gives

$$\ddot{R} = -\frac{2}{9} \beta t^{-4/3}. \tag{4.13}$$

Substituting equ(4.11)(18), equ(4.12)(18) and equ(4.13)(18) into equ(2.11)(9) yields

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = -\frac{\left(\beta t^{2/3}\right)\left(-\frac{2}{9}\beta t^{-4/3}\right)}{\left(\frac{2}{3}\beta t^{-1/3}\right)\left(\frac{2}{3}\beta t^{-1/3}\right)} = \frac{\frac{2}{9}\beta^2 t^{-2/3}}{\frac{4}{9}\beta^2 t^{-2/3}} = 0.5,$$

which agrees with the standard textbook value of the deceleration parameter for the Einstein-de Sitter universe.

Another way to evaluate the deceleration parameter is to equate equ(3.7)(14) and equ(3.8)(14), since $\lambda = 0$ and $K = 0$, giving

$$4\pi G\rho - 3H^2q = 4\pi G\rho - H^2(q+1).$$

Subtracting $4\pi G\rho$ from both sides, then dividing both sides by H^2 gives

$$3q = q + 1, \text{ or } q = 0.5.$$

Another way to solve for the deceleration parameter substitutes $K = 0$ into equ(4.1)(15), giving

$$0 = H^2(2q - 1).$$

Therefore, either $H = 0$, which is impossible in the Einstein-de Sitter universe where $\dot{R}$ never reaches 0 (the expansion never quite stops), or $2q - 1 = 0$, giving $q = 0.5$.

One last way to solve for the deceleration parameter, just for good luck, substitutes $\lambda = 0$ into equ(3.7)(14) giving

$$4\pi G\rho = 3H^2q.$$

Rearranging this gives

$$\rho = \frac{3H^2}{4\pi G}q,$$

which you might recognize as equ(4.3)(16). Substituting $k = 0$ and $\lambda = 0$ into equ(3.2)(12), we obtain

$$H^2 = \frac{8\pi}{3}G\rho,$$

and substitute that equation into the right side of the one above to give

$$\rho = \frac{8\pi G\rho}{4\pi G}q = 2\rho q,$$

proving that $q = 0.5$. This seemingly endless set of diversions into the deceleration parameter was simply to show that this is a self-consistent theory, and we'd better get the same answer no matter which set of equations we use.

Now we will turn our attention to finding the present-day critical density, ρ_c, based on our present-day best-estimates of the Hubble Parameter. This critical density is the density the universe must have if it fits the Einstein-de Sitter model.

Evaluating equ(4.3)(16) with $q_0 = 0.5$, at $H_0 = 50$ km s^{-1} Mpc^{-1} $= 1.6 \times 10^{-18}$ s^{-1} gives

$$\rho_c = \frac{3(1.6 \times 10^{-18}\ \mathrm{s}^{-1})(1.6 \times 10^{-18}\ \mathrm{s}^{-1})}{8\pi(6.67 \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2})} = 4.58 \times 10^{-27}\frac{\mathrm{kg}}{\mathrm{m}^3} \tag{4.14}$$

or, assuming 1.66×10^{-24} grams/atom,

$$\rho_c = \left(4.58 \times 10^{-27}\frac{\mathrm{kg}}{\mathrm{m}^3}\right)\left(\frac{1\ \mathrm{atom}}{1.66 \times 10^{-27}\ \mathrm{kg}}\right) = 2.76\frac{\mathrm{atoms}}{\mathrm{m}^3}.$$

Evaluating equ(4.3)(16) at $H_0 = 100$ km s^{-1} Mpc^{-1} $= 3.2 \times 10^{-18}$ s^{-1} gives

$$\rho_c = \frac{3(3.2 \times 10^{-18}\ \mathrm{s}^{-1})(3.2 \times 10^{-18}\ \mathrm{s}^{-1})}{8\pi(6.67 \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2})} = 1.83 \times 10^{-26}\frac{\mathrm{kg}}{\mathrm{m}^3} \tag{4.15}$$

or, assuming 1.66×10^{-24} grams/atom,

$$\rho_c = \left(1.83 \times 10^{-26}\frac{\text{kg}}{\text{m}^3}\right)\left(\frac{1 \text{ atom}}{1.66 \times 10^{-27} \text{ kg}}\right) = 11.0\frac{\text{atoms}}{\text{m}^3}.$$

So, if our Universe fits the Einstein-de Sitter model, and $50 <= H_0 <= 100$, then there are between 3 and 11 atoms per cubic meter.

To check our work, we'll try to come up with the critical density in a different way by substituting equ(2.9)(9) into equ(4.7)(17) to obtain

$$R^3 = 6\pi G(\rho R^3)t^2,$$

or

$$\rho = \frac{1}{6\pi G t^2}.$$

Evaluating this equation at the time given by equ(4.9)(18) gives

$$\rho = \frac{1}{6\pi(6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2})(4.17 \times 10^{17} \text{ s})(4.17 \times 10^{17} \text{ s})} = 4.57 \times 10^{-27}\frac{\text{kg}}{\text{m}^3}$$

which agrees with equ(4.14)(19). Evaluating at the time given in equ(4.10)(18) gives

$$\rho = \frac{1}{6\pi(6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2})(2.08 \times 10^{17} \text{ s})(2.08 \times 10^{17} \text{ s})} = 1.84 \times 10^{-26}\frac{\text{kg}}{\text{m}^3}$$

which agrees with equ(4.15)(19).

To evaluate the density parameter in the Einstein-de Sitter model, solve equ(3.1)(12) for ρ and substitute that into the numerator of equ(2.13)(10) and equ(2.10)(9) into the denominator, giving

$$\Omega = \frac{\rho}{\rho_c} = \frac{3H^2/8\pi G}{3H^2/8\pi G} = 1.$$

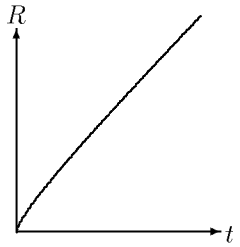
As we found in equ(4.4)(16), $\Omega = 2q$, which equals 1 since $q = 0.5$.

The Einstein-de Sitter model is always an accurate description of a Big Bang modelled with the Friedmann equation at early times in the matter-dominated era[2], regardless of the value of λ or k. Recalling equ(3.4)(12),

$$\dot{R}^2 = \frac{8\pi}{3}G\rho_0 R^{-1} + \frac{\lambda}{3}R^2 - k,$$

we can see that if R is extremely small, the gravity term overwhelms both the lambda term and the curvature term, so only the gravity term is important at very early times. **Therefore, equ(4.7)(17) is valid at early times in any model which begins with expansion from R(0)=0.**

[2] See Silk (1980) page 334 mathematical note 6.



MODEL# 2

Open Friedmann-Lemâitre

$k < 0$; $\lambda = 0$; $R(0) = 0 \Uparrow$; BW; Harrison 3,E; Bondi 1ii

We begin our analysis of this model by substituting $\lambda = 0$ into equ(3.4)(12) (the Friedmann equation), giving

$$\dot{R}^2 = \frac{8\pi}{3} G\rho_0 R^{-1} - k.$$

Since $k < 0$ in this model, $\dot{R}^2$ is the sum of two positive terms, the gravity term and the curvature term. Consequently, however large or small R becomes, $\dot{R}^2$ can never vanish. This means that the scaling factor will increase without limits, forever. This universe ends in an open state with a whimper. When the universe is young, this model behaves like model#1, equ(4.7)(17). As R grows, the gravity term gets smaller and the curvature term remains constant, therefore $\dot{R}^2$ gets continuously smaller. The rate of expansion continually decreases, but never stops. When R is very large, the gravity term will become vanishingly small, and we can approximate the universe with

$$\dot{R}^2 \approx -k.$$

Taking the square root of both sides gives

$$\dot{R} \approx \sqrt{-k}.$$

We can integrate this with respect to time using equ(1.3)(5) for the left side and equ(1.2)(5) for the right side giving

$$R \approx \sqrt{-k}\ t + (\text{some constant of integration}),$$

which shows that the scale factor of the universe grows without limit as a function of time.

Since the universe has negative curvature ($k < 0$), it is said to have "hyperbolic geometry". The universe does not have enough mass to stop the expansion, so its density must be less than the critical density. At any given time past the Big Bang, this universe has a larger scaling factor and a larger Hubble Parameter than the Einstein-de Sitter universe (model#1) would at the same time. Any given value of the Hubble Parameter is reached longer after the Big Bang than in an Einstein-de Sitter (model#1) universe. Therefore the age of the universe under this model is greater than $\frac{2}{3}\tau$, which was the age we came up with for an Einstein-de Sitter (model#1) universe in equ(4.8)(18).

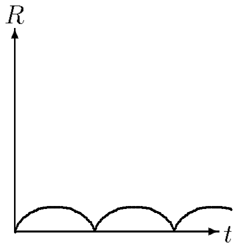
To get a handle on the deceleration parameter, substitute equ(2.15)(10) into equ(4.1)(15) giving

$$kR^{-2} = H^2(2q - 1).$$

Substituting what little information we have, we can determine that

$$\text{negative} = \text{positive} \times (2\text{q} - 1),$$

therefore $2q - 1 < 0$, and $q < 0.5$.



MODEL# 3

Closed Friedmann-Lemâitre

$k > 0$; $\lambda = 0$; $R(0) = 0 \Uparrow$; BB; Harrison 1,J; Bondi 3iv

We begin again by substituting $\lambda = 0$ into equ(3.4)(12)

$$\dot{R}^2 = \frac{8\pi}{3} G\rho_0 R^{-1} - k.$$

Since $k > 0$, $\dot{R}^2$ is the sum of a positive gravity term and a negative curvature term. When R is extremely small, the gravity term is extremely large, therefore the universe expands rapidly, behaving the same as model#1, equ(4.7)(17). As R increases, the gravity term becomes smaller, and the expansion of the universe decelerates. Eventually there will be a time when the curvature term exactly balances the gravity term, and $\dot{R}^2$ will become zero; the universe will stop expanding. Now one of two things can happen according to the equation. The first is that the universe begins its expansion again. However, if this happens, the gravity term is smaller still, and $\dot{R}^2$ becomes negative. This is a mathematical impossibility, so this option cannot be correct. The other thing that can (and does) happen, is that the universe begins contracting, ever so slowly at first. The gravity term then becomes slightly larger than the curvature term again, and the contraction accelerates. The smaller R becomes, the faster the contraction progresses since the gravity term grows as the inverse of R. The universe (or at least this cycle of it) ends with a bang and is said to be closed. Because $k > 1$, the geometry of space is said to be "spherical". No one knows what happens when R reaches zero, however many authors call this an oscillating universe, suggesting that a new cycle begins whenever one comes to an end. Since the gravitational pull of the matter in the universe was large enough to stop the expansion, the density of that matter must have been greater than the critical density. At any given time past the Big Bang, this universe has a smaller scaling factor and a smaller Hubble Parameter than the Einstein-de Sitter (model#1) universe at the same time. Any given value of the Hubble Parameter is reached sooner after the Big Bang than in an Einstein-de Sitter (model#1) universe. Therefore the age of the universe under this model is less than $\frac{2}{3}\tau$, which was the age we came up with for an Einstein-de Sitter (model#1) universe in equ(4.8)(18). This model is similar to model#7, model#14, model#17, model#22, and model#23. Harrison's low-h high-Ω model[3] uses this model, with $H_0 = 10$ km s^{-1} Mpc^{-1} and $\Omega = 10$.

To determine the deceleration parameter in this model, substitute equ(2.15)(10) into equ(4.1)(15)

$$kR^{-2} = H^2(2q - 1).$$

We again use a qualitative argument, substituting what little information we have, giving

$$\text{positive} = \text{positive} \times (2\text{q} - 1),$$

therefore $2q - 1 > 0$, and $q > 0.5$.

We can determine the scaling factor when the expansion changes into contraction. It is the value of the R when $\dot{R} = 0$. Recalling equ(3.4)(12) with $\lambda = 0$,

$$\dot{R}^2 = \frac{8\pi}{3} G\rho_0 R^{-1} - k,$$

and substituting the condition we are deriving yields

$$0 = \frac{8\pi}{3} G\rho_0 R_{max}^{-1} - k,$$

[3] See Harrison (1993a), and Harrison (1993b).

which rearranges into

$$R_{max} = \frac{8\pi}{3k} G\rho_0.$$

The three $\lambda = 0$ models we have just finished studying are summarized in table 4.1 and the age of those three universes is shown in figure 4.1.

	Model#2	Model#1	Model# 3
name	open Friedmann-Lemâitre	Einstein de-Sitter	closed Friedmann-Lemâitre
k	< 0	$= 0$	> 0
q	< 0.5	$= 0.5$	> 0.5
universe	open	open	closed
geometry	hyperbolic	flat	spherical
curvature	negative	none	positive
age	$> (2/3)\tau$	$= (2/3)\tau$	$< (2/3)\tau$
density	$<$ critical	$=$ critical	$>$ critical
beginning	bang	bang	bang
end	whimper	whimper	bang

Table 4.1: Summary of the $\lambda = 0$ models

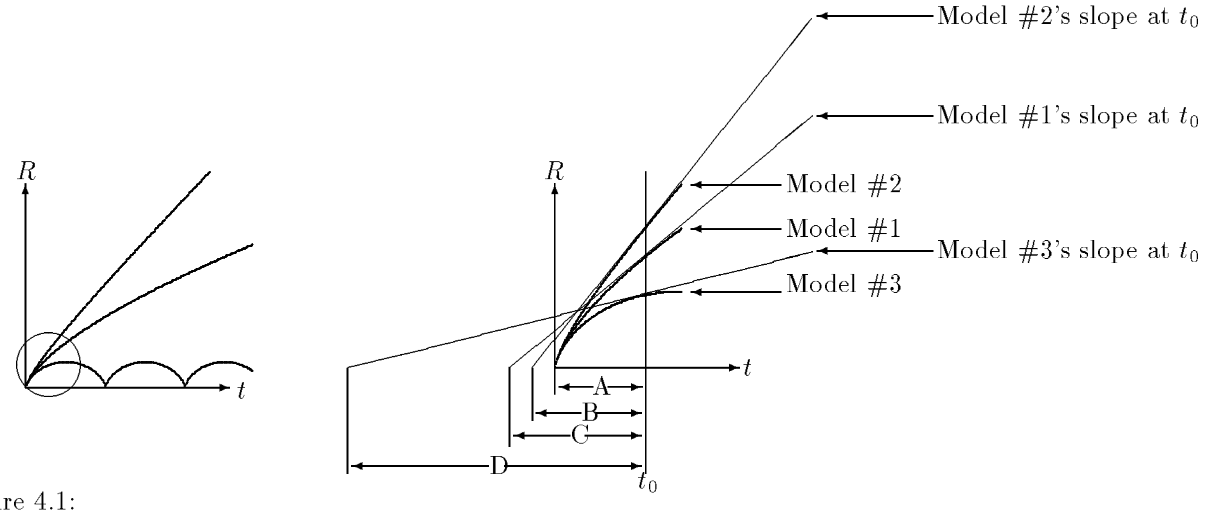
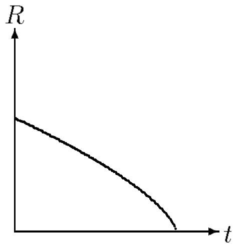


Figure 4.1:
τ and the age of the universe in the $\lambda = 0$ models. The graph on the left shows all 3 models. The circled region is expanded in the graph on the right. A vertical line is draw at some time which might represent the time at which we now live (t_0). At that time, the slope of each of the models is drawn. This line represents the Hubble Parameter (H_0) we would measure at time t_0. The slopes are extrapolated backward in time to the $R = 0$ line, and 4 ages of the universe are labeled below the graph. Age A is the true age of the universe in all 3 models. Age B is τ measured at time t_0 in model#2. The true age of universe #2 is $> 2/3\tau$. Age C is τ measured at time t_0 in model#1. The true age of universe #1 is $= 2/3\tau$ (see equ(4.8)(18)). Age D is τ measured at time t_0 in model#3. The true age of universe #1 is $< 2/3\tau$.

We now will examine the other two $\lambda = 0$ models, the ones which are not widely-accepted.

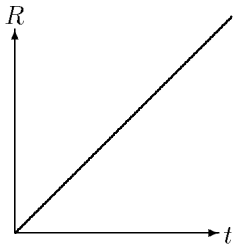


MODEL# 4

Crunch

$k = 0$; $\lambda = 0$; $R(0) > 0 \Downarrow$; WB; Harrison 10; Bondi -

This model appears to have nothing to do with reality, however it *is* a model based on the Friedmann equation with $\lambda = 0$, so is included here for completeness. The universe begins with a nonzero scaling factor and contracts until the Big Crunch is reached. This model is simply a time reversal of model#1.



MODEL# 5

Milne Model, or Kinematic Relativity

$k < 0$; $\lambda = 0$; $R(0) = 0 \Uparrow$; BW; Harrison -; Bondi -

This is an unorthodox cosmological model which assumes that gravity does not exist[4]. The universe starts out with an explosion and does not decelerate. $G = 0$, $\lambda = 0$, $\dot{R}$ is constant, and $q = 0$, which leaves us with a Friedmann equation of $H^2 = -kR^{-2}$ (from equ(3.2)(12)), or $\dot{R}^2 = -k$ (from equ(3.3)(12)). The former of these two equations can be rearranged to

$$(-k)^{-1/2}R = H^{-1}. \tag{4.16}$$

Taking the square root of both sides of the latter of the two equations, we obtain

$$\dot{R} = (-k)^{1/2}.$$

Using equ(1.3)(5) for the left side and equ(1.2)(5) for the right side, we can integrate this, giving

$$R = (-k)^{1/2}t,$$

which tells us the scaling factor of this universe at any point in time. Obviously, it is a linear function. We can also rearrange this equation to obtain

$$t = (-k)^{-1/2}R.$$

Substituting equ(4.16)(24) into this equation yields

$$t = H^{-1} = \tau,$$

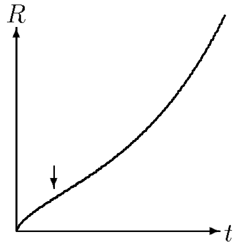
which tells us that the age of the universe at any time is equal to the Hubble Period.

[4]See Harrison (1981) pages 316-318, Bondi (1952) pages 123-139, or Peebles (1993) §7 for more information on this model.

Chapter 5

Friedmann Models with $\lambda \neq 0$

This chapter examines the Friedmann Equation for models which allow a nonzero cosmological constant. This opens-up a multitude of other possibilities for the history and future of the universe.



MODEL# 6

Lambda

$k = 0$; $\lambda > 0$; $R(0) = 0 \Uparrow$; BW; Harrison A; Bondi 2i

For our first excursion into the cosmological constant, we'll look at the easiest model which uses it. This model is the same as model#1, except a positive cosmological constant is added. Beginning by setting $k = 0$ into the equ(3.4)(12) version of the Friedmann equation, we obtain

$$\dot{R}^2 = \frac{8\pi}{3} G\rho_0 R^{-1} + \frac{\lambda}{3} R^2.$$

The first thing we notice is that the rate of change of R is the sum of 2 positive terms, so there is no negative term to stop the expansion. This is an open universe that expands without limit. At early times, the model is the same as the Einstein-de Sitter model (model#1), however, a positive cosmological constant acts as a repulsive force in the universe, which makes the universe expand faster than it would without the so-called lambda-force.

$\dot{R}$, the rate of change of R, is at a minimum when $\ddot{R} = 0$. To find when the model reaches this point (it happens somewhere near the arrow I drew into the diagram) we substitute $\ddot{R} = 0$ into equ(3.5)(13), obtaining

$$\frac{4\pi}{3} G\rho R = \frac{\lambda}{3} R.$$

Multiply both sides of the equation by $3/R$ and substitute equ(2.9)(9) into here to obtain

$$4\pi G\rho_0 R^{-3} = \lambda,$$

and solving for R gives

$$R = \sqrt[3]{\frac{\lambda}{4\pi G\rho_0}}.$$

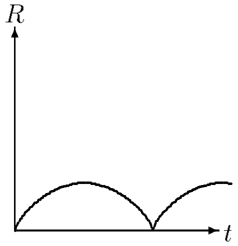
This is the value of the scaling factor when the rate of change of the scaling factor is least.

Bondi (1952) does not show the derivation, and I cannot figure one out, but he states that it can be shown that this case has the explicit solution of

$$R^3 = \frac{4\pi G\rho_0}{\lambda}[\cosh(t\sqrt{3\lambda}) - 1].$$

To determine the age of a universe which fits this model, you must know the value of λ as well as the Hubble Parameter.

This model is similar to models #13 and #16.



MODEL# 7

Closed

$k = 0$; $\lambda < 0$; $R(0) = 0 \Uparrow$; BB; Harrison C,1; Bondi 2iii

Just as a positive lambda term tends to counteract gravity, a negative lambda term reinforces gravity. Compare this model to the models we've already studied and you'll see that it behaves like a model#1 universe with more mass, or a model#3 (positive curvature) universe.

The Friedmann equation, with $k = 0$, $\lambda < 0$, substituted into equ(3.4)(12) looks like this

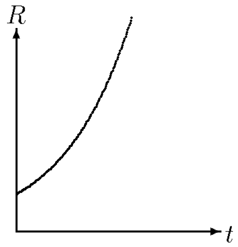
$$\dot{R}^2 = \frac{8\pi}{3}G\rho_0 R^{-1} + \frac{\lambda}{3}R^2.$$

This equation states that the rate of change of the scaling factor is the sum of a positive gravity term and a negative lambda term. When R is extremely small, the gravity term is very large and the lambda term is very small, so the gravity term dominates the drive to expand the universe, exactly as it does in model#1, equ(4.7)(17). As R grows, the gravity term becomes a smaller positive number and the lambda term becomes a larger negative number, slowing the expansion. Eventually at some R, the two terms balance each other out and the universe stops expanding. Three things can happen now, looking at the equation and the diagram. The first is that the universe can remain at the same scaling factor, resulting in a static universe. This may indeed happen for a short time, but any higher-than-average-density region will drive the universe away from the static state. The second thing that might happen is that the universe begins expanding again, but a careful look at the equation shows that this cannot happen. If it did, then the lambda term would be greater than the gravity term, the sum would be negative, and the universe wouldn't be able to perform the required square root function. The only thing that can really happen is for the universe to begin contracting, ever so slowly at first since the gravity term will be just slightly larger than the lambda term. As the contraction progresses, though, it accelerates as gravity dominates the lambda term by a wider and wider margin, resulting in a Big Crunch.

Again, Bondi (1952) does not show the derivation, and I cannot figure one out, but he states that it can be shown that this case has the explicit solution of

$$R^3 = -\frac{4\pi G\rho_0}{\lambda}[1 - \cos(t\sqrt{-3\lambda})].$$

This model is similar to models #3, #14, #17, #22, and #23.



MODEL# 8

Expanding de Sitter

$k = 0$; $\lambda = 3H^2$; $R(0) > 0 \Uparrow$; WW; Harrison 9_1; Bondi -

This was one of the earliest models that was considered when the modern science of cosmology was in its infancy. There is no matter in this model, and space is flat. However, space still expands because of the lambda force. The universe is "born" with a nonzero scaling factor. Substituting $\rho = 0$ and $k = 0$ into the Friedmann equation (equ(3.2)(12))gives

$$H^2 = \frac{\lambda}{3}. \tag{5.1}$$

Notice that rearranging this equation gives the value for λ in the model heading line. This equation states that the repulsive lambda force causes space to expand with constant acceleration. After taking the square root of both sides and substituting equ(2.4)(8), this equation becomes

$$\frac{1}{R}\dot{R} = \sqrt{\frac{\lambda}{3}}.$$

As a preparatory step to allow for integration of this equation, we will force the left side of this equation to conform to the form of the right side of equ(1.8)(5) by using $a = e$ (so that we are multiplying it by $log_a e = 1$), giving

$$(log_e e)\frac{1}{R}\dot{R} = \sqrt{\frac{\lambda}{3}},$$

which by use of equ(1.8)(5) turns into

$$\frac{d}{dt}(log_e R) = \sqrt{\frac{\lambda}{3}}.$$

We can integrate both sides (using equ(1.2)(5) for the right side) to get

$$log_e R = \sqrt{\frac{\lambda}{3}}t.$$

Taking the exponential of both sides of this equation gives

$$R = e^{((\lambda/3)^{1/2}t)}, \tag{5.2}$$

which gives us the scaling factor of the universe at any time.

To determine the value of the deceleration parameter in this model, substitute $\rho = 0$ and $k = 0$ into equ(3.8)(14), giving

$$0 = 0 - H^2(q+1),$$

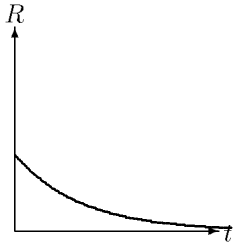
which implies that either $H^2 = 0$ (which is impossible since $\lambda = 3H^2 \neq 0$), or $q + 1 = 0$, which implies that

$$q = -1.$$

Recall that a deceleration parameter less than 0.5 results in an ever-expanding open universe. Just to show a bit of consistency in the equations, substitute $\rho = 0$ and $q = -1$ into equ(3.7)(14) to obtain

$$\lambda = 0 - 3H^2(-1) = 3H^2,$$

which agrees with equ(5.1)(27).

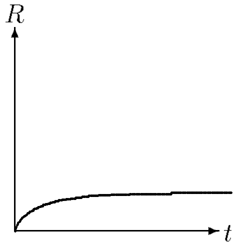


MODEL# 9

Contracting de Sitter

$k = 0$; $\lambda = 3H^2$; $R(0) > 0 \Downarrow$; WW; Harrison 9_2; Bondi -

This model is the same as model#8, except it begins in a contracting state rather than an expanding state. It is a time-reversal of model#8.



MODEL# 10

Eddington-Lemâitre

$k > 0$; $\lambda = \lambda_E$; $R(0) = 0 \Uparrow$; BS; Harrison $2, H_1$; Bondi 3iib

The combination of positive curvature and positive lambda force puts the universe into a tug of war. The curvature tries to close the universe, while the lambda force tries to expand it, or open it. There is a critical value of λ which makes the lambda force perfectly balance out the curvature term and results in a universe which expands from a Big Bang to some scaling factor, then becomes static. For this model, that critical value of λ, which is called λ_E, is equ(2.16)(10), which we'll derive in model # 12,

$$\lambda_E = \frac{k^3}{(4\pi G\rho_0)^2}$$

and substituting this value into the equ(3.4)(12) version of the Friedmann equation gives

$$\dot{R}^2 = \frac{8\pi}{3}G\rho_0 R^{-1} + \frac{k^3}{3(4\pi G\rho_0)^2}R^2 - k. \tag{5.3}$$

At early times, R is very small so the gravity term is very large, the lambda term is very small, and the curvature term is small. We are left with the approximation

$$\dot{R}^2 \approx \frac{8\pi}{3}G\rho_0 R^{-1},$$

which was analyzed under model#1 and resulted in resulted in equ(4.7)(17). As time passes, the scale factor grows, making the gravity term smaller, the lambda term larger, and the curvature term's contribution larger, relative to what it used to be, thereby slowing the rate of expansion. Remembering that we have carefully chosen the value of λ such that it perfectly balances the gravity term and the curvature term, it should be evident that the rate of expansion will slow to zero.

We will now find the scale factor's value when the rate of change of the scale factor becomes zero. For convenience we will define the constant

$$\beta = \frac{8\pi}{3}G\rho_0,$$

and set $\dot{R}$ to zero in equ(5.3)(28), giving

$$0 = \beta R^{-1} + \frac{4k^3}{27\beta^2} R^2 - k.$$

Multiplying both sides by $\frac{27\beta^2 R}{4k^3}$ gives

$$0 = \frac{27\beta^3}{4k^3} + R^3 + \frac{27\beta^2}{4k^2} R,$$

which we can rearrange into

$$0 = R^3 - \frac{27\beta^2}{4k^2} R + \frac{27\beta^3}{4k^3},$$

which factors into

$$0 = \left(R - \frac{3\beta}{2k}\right)\left(R^2 + \frac{3\beta}{2k} R - \frac{9\beta^2}{2k^2}\right) = \left(R - \frac{3\beta}{2k}\right)\left(R - \frac{3\beta}{2k}\right)\left(R + \frac{3\beta}{k}\right).$$

Since one of these three factors must be zero, either $R = \frac{3\beta}{2k}$ or $R = -\frac{3\beta}{k}$. The latter makes no sense, so we choose the former and replace our convenient constant's value back in, giving

$$R = \frac{4\pi G\rho_0}{k}.$$

A much simpler way to derive this is to note that when $\dot{R} = 0$, $H = 0$. Substituting $H = 0$ into equ(3.8)(14) gives

$$K = 4\pi G\rho.$$

Substituting equ(2.15)(10) and equ(2.9)(9) into here gives

$$kR^{-2} = 4\pi G\rho_0 R^{-3},$$

which rearranges into

$$R = \frac{4\pi G\rho_0}{k}. \tag{5.4}$$

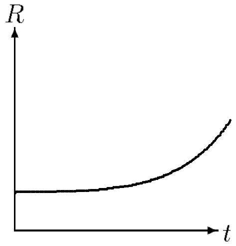
This is the value of the scaling factor when the universe stops expanding. To check our work, let us substitute that value back into equ(5.3)(28)

$$\dot{R}^2 = \frac{8\pi}{3} G\rho_0 \left(\frac{k}{4\pi G\rho_0}\right) + \frac{k^3}{3(4\pi G\rho_0)^2}\left(\frac{4\pi G\rho_0}{k}\right)^2 - k,$$

which easily simplifies down to

$$\dot{R}^2 = \frac{2}{3}k + \frac{1}{3}k - k = 0,$$

proving we came up with the correct value for R when the universe stops expanding.



MODEL# 11

Eddington-Lemâitre

$k > 0$; $\lambda = \lambda_E$; $R(0) > 0 \Rightarrow$; SW; Harrison 8,H_2; Bondi 3iic

This is the time-reversal of model#10. R begins at the static value we calculated for model#10. Any area of increased density can cause the gravity term to become slightly larger than the curvature constant can balance, at least in some local area, and the universe expands. When that happens, the carefully-balanced lambda term falls to pieces, causing runaway expansion. It should be noted that model#10 can end with this same runaway expansion, since its static state is also very tentative. Once the runaway expansion of the universe begins, the gravity term becomes negligible (the falling density is divided by an ever-increasing scaling factor), the curvature term, which is constant, becomes less and less in relation to the lambda term, and we are left with the approximation

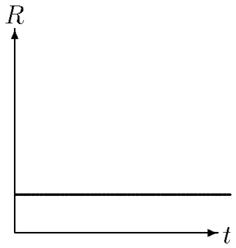
$$\dot{R}^2 = \frac{\lambda}{3} R^2,$$

which is easily transformed via equ(2.4)(8) into

$$H^2 = \frac{\lambda}{3}.$$

This approximation was analyzed in model#8 to yield equ(5.2)(27)

$$R = e^{((\lambda/3)^{1/2} t)}.$$



MODEL# 12

Einstein

$k > 0$; $\lambda = \lambda_E$; $R(0) > 0 \Rightarrow$; S; Harrison 6; Bondi 3iia

Precariously balanced between model#10 and model#11 we find Einstein's original model of a static universe. This universe, of course, neither expands nor contracts, which means $H = 0$ and $\dot{R} = 0$. Substituting $H = 0$ into equ(3.8)(14) leaves

$$K = 4\pi G\rho.$$

Substituting equ(2.15)(10) into here gives

$$kR^{-2} = 4\pi G\rho,$$

which we can rearrange into

$$R^2 = \frac{k}{4\pi G\rho}. \tag{5.5}$$

Substituting equ(2.9)(9) into here gives

$$R^2 = \frac{k}{4\pi G\rho_0 R^{-3}},$$

which rearranges into

$$R = \frac{4\pi G\rho_0}{k}, \tag{5.6}$$

which gives us the scaling factor of the Einstein universe. Notice that it is a constant, and that it matches equ(5.4)(29).

To find the value of λ, substitute $H = 0$ and equ(2.9)(9) into equ(3.2)(12), the Friedmann equation

$$0 = \frac{8\pi}{3}G\rho_0 R^{-3} + \frac{\lambda_E}{3} - kR^{-2},$$

and substitute our value for R from equ(5.5)(30), giving

$$0 = \frac{2}{3}(4\pi G\rho_0)\left(\frac{k^3}{(4\pi G\rho_0)^3}\right) + \frac{\lambda_E}{3} - k\left(\frac{k^2}{(4\pi G\rho_0)^2}\right),$$

which simplifies to

$$0 = \frac{2}{3}\left(\frac{k^3}{(4\pi G\rho_0)^2}\right) + \frac{\lambda_E}{3} - 1\left(\frac{k^3}{(4\pi G\rho_0)^2}\right) = \frac{\lambda_E}{3} - \frac{1}{3}\left(\frac{k^3}{(4\pi G\rho_0)^2}\right).$$

Multiplying by 3 and rearranging yields our final answer of

$$\lambda_E = \frac{k^3}{(4\pi G\rho_0)^2}, \tag{5.7}$$

which agrees with our definition of λ_E in equ(2.16)(10).

We can eliminate the curvature constant from λ_E through the following manipulations. Begin by substituting equ(2.9)(9) into our equation

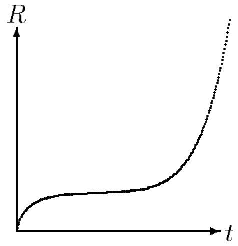
$$\lambda_E = \frac{k^3}{(4\pi G\rho R^3)^2} = \left(\frac{k^3}{(4\pi G\rho)^2}\right)R^{-6},$$

then substituting equ(5.5)(30)

$$\lambda_E = \left(\frac{k^3}{(4\pi G\rho)^2}\right)\left(\frac{(4\pi G\rho)^3}{k^3}\right) = 4\pi G\rho, \tag{5.8}$$

which matches equ(2.17)(10), and also matches the result one would get by substituting $H = 0$ into equ(3.7)(14), except without all the fun we've had going the long way around.

Since $K = 4\pi G\rho$ and $\lambda_E = 4\pi G\rho$, $\lambda_E = K$, so the universe has positive curvature (is said to be spherical) equal to the cosmological constant.



MODEL# 13

Lemaître

$k > 0$; $\lambda > \lambda_E$; $R(0) = 0 \Uparrow$; BSW; Harrison 5,G; Bondi 3i

Beginning with the equ(3.4)(12) version of the Friedmann Equation,

$$\dot{R}^2 = \frac{8\pi}{3}G\rho_0 R^{-1} + \frac{\lambda}{3}R^2 - k,$$

we find that we cannot eliminate any terms of the equation in this model. However I think we can still learn a bit about it through some qualitative arguments. Recalling our argument at the bottom of the discussion of model#1, this model must begin as that model begins, with a standard Big Bang. It must follow equ(4.7)(17) at early times. Soon after the early phase of the universe, though, the curvature term becomes large enough compared to the other two terms to slow the expansion, trying to force it into a model#3. At some stage, R becomes large enough for the lambda force of cosmic repulsion to exert its influence, counteracting the curvature term's desire to contract the universe, thereby forcing the universe into a coasting phase. But notice that the curvature term gets no boost as R grows, but the lambda term does. The boost which lets the lambda term eventually overwhelm the curvature term is provided by the gravity term which is contributing less and less to the expansion as R increases, but is nevertheless giving enough of a gentle nudge to assure that the lambda force wins. Once the lambda term is larger than the curvature term, nothing can stop it from generating runaway universal expansion.

Now for a bit of math... $\dot{R}$ is always positive in the equation for this model, with a minimum at a value of R which we will now derive. That is, we're going to find out the value of R when the universe is expanding at the slowest rate. During the coasting phase, $\dot{R} = 0$, so

$$0 = \frac{8\pi}{3} G\rho_0 R^{-1} + \frac{\lambda}{3} R^2 - k.$$

Substituting equ(2.15)(10) and multiplying both sides by 3 we obtain

$$0 = 8\pi G\rho_0 R^{-1} + \lambda R^2 - 3KR^2.$$

Substituting equ(3.8)(14) with $H^2 = 0$ since $\dot{R} = 0$ gives

$$0 = 8\pi G\rho_0 R^{-1} + \lambda R^2 - 3(4\pi G\rho)R^2.$$

Substituting equ(2.9)(9) gives

$$0 = 8\pi G\rho_0 R^{-1} + \lambda R^2 - 12\pi G\rho_0 R^{-1}.$$

Subtracting λR^2 from both sides, then dividing both sides by $-R^{-1}$ gives

$$\lambda R^3 = 12\pi G\rho_0 - 8\pi G\rho_0 = 4\pi G\rho_0, \tag{5.9}$$

or

$$R = \sqrt[3]{\frac{4\pi G\rho_0}{\lambda}}, \tag{5.10}$$

which gives us the scaling factor during the coasting phase. If instead we solve equ(5.9)(32) for λ, we obtain

$$\lambda = 4\pi G\rho_0 R^{-3}.$$

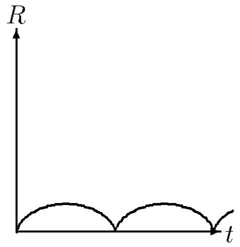
Substituting equ(2.9)(9) we obtain

$$\lambda = 4\pi G\rho,$$

which is λ_E. However, since $\lambda > \lambda_E$ in this model, the coasting phase does not last, and the lambda term causes runaway expansion.

This universe has often seemed attractive to cosmologists, because the coasting period provides a plausible era when the condensation of galaxies could occur. Because light might circumnavigate the universe more than once during a long hesitation era, and no "ghost" images are observed, the model is not seriously considered today.

This model is similar to models #6 and #16.



MODEL# 14

Closed

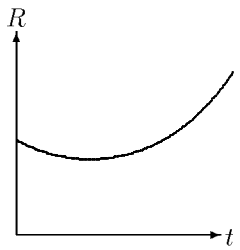
$k > 0$; $0 < \lambda < \lambda_E$; $R(0) = 0 \Uparrow$; BB; Harrison 1,I_1; Bondi 3iiia

Referring to the equ(3.4)(12) version of the Friedmann Equation

$$\dot{R}^2 = \frac{8\pi}{3} G\rho_0 R^{-1} + \frac{\lambda}{3} R^2 - k.$$

$\dot{R}$ is the sum of two positive terms (gravity and lambda) and one negative curvature term. The universe begins exactly as does model#1, equ(4.7)(17). We know, from model#12, that if $k > 0$ and $\lambda = \lambda_E$, we have a static universe with $\dot{R} = 0$. We also know, from model#3, that if $k > 0$ and $\lambda = 0$, we have a closed universe. Now we are looking at the intermediate case where $k > 0$ and $0 < \lambda < \lambda_E$. This means that the lambda term, which tends to force the universe to expand, is not strong enough to counteract the curvature constant, which tends to force the universe to contract. Therefore the universe closes. We can also make this argument mathematically, by examining the equation, rather than just by comparing it against other models we've already examined. When the universe reaches the static instant, that is, when $\dot{R} = 0$, if the curvature term becomes dominant, the right side of the equation would become negative, which is illegal. We know that the lambda term cannot be dominant, because it is less than the critical value which keeps the universe static. So the gravity term must be dominating the equation. We still don't know what the gravity constant is tending to do to the rate of change of the scaling factor at this point, though. It's not going to hold the universe static, because $\lambda = \lambda_E$ would be required for that. It's not going to expand the universe more, because the curvature constant was large enough to counter it, so the gravity term must force contraction at this point in the universe' evolution. Once the contraction begins, the gravity term continues to dominate because it is getting larger with smaller R, and the lambda term is getting smaller with smaller R. The universe must continue contracting until the Big Crunch.

This model is similar to models #3, #7, #17, #22 and #23.



MODEL# 15

Bounce

$k > 0$; $0 < \lambda < \lambda_E$; $R(0) > 0 \Downarrow$; WW; Harrison 12,I_2; Bondi 3iiib

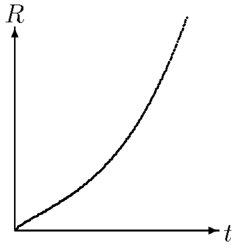
Notice that the parameters for this model are identical to the parameters for model#14, except the universe starts with a nonzero scaling factor and begins by contracting. Referring to the equ(3.4)(12) version of the Friedmann Equation

$$\dot{R}^2 = \frac{8\pi}{3} G\rho_0 R^{-1} + \frac{\lambda}{3} R^2 - k.$$

At an early stage, the gravity term is negligible due to the large scaling factor, so the lambda term and the curvature term must be controlling the rate of change of the scaling factor. When the scaling factor is very large, the lambda term is a very large positive number, the positive curvature term is subtracted and the rate of change is large. This results in a smaller scaling factor, which decreases the lambda term's excess over the curvature term. The rate of change of the scaling factor declines until it eventually becomes zero. Note that the scaling factor does NOT have the same value at this point as it did at the static moment in model#14. There is a band of scaling factor values at which the right side of the equation is negative, therefore are prohibited. Since the universe cannot contract into this forbidden band, it must either remain static or expand again. Since $\lambda \neq \lambda_E$, we know it cannot remain static and must expand. Once the expansion begins, the lambda force grows rapidly as the scaling factor increases, and the expansion accelerates.

At times after the static instant, the same arguments as used under model#11 are valid, with the result that the scaling factor is the exponential function of time given in equ(5.2)(27). Before the time of the static instant, the scaling factor is simply a time-reversal of that given in equ(5.2)(27), namely

$$R = e^{((\lambda/3)^{1/2}t)}.$$



MODEL# 16

Delayed

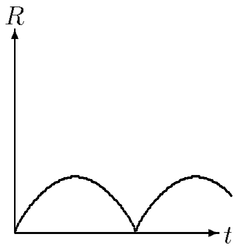
$k < 0$; $\lambda > 0$; $R(0) = 0 \Uparrow$; BW; Harrison D; Bondi 1i

Referring to the equ(3.4)(12) version of the Friedmann Equation

$$\dot{R}^2 = \frac{8\pi}{3}G\rho_0 R^{-1} + \frac{\lambda}{3}R^2 - k.$$

With the values given for this model, the rate of change of the scaling factor is the sum of three positive terms, and like model#6, we have eternal expansion. In its early stages, when R is extremely small, the gravity term dominates and this model must behave like model#1, equ(4.7)(17). The initial rate of expansion slows as in model#13, until equ(5.10)(32) is reached. Then, at large R, the lambda term dominates and the rate of expansion increases like model#8, equ(5.2)(27).

This model is similar to models #6 and #13.



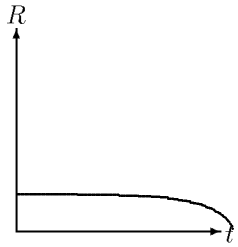
MODEL# 17

Closed

$k < 0$; $\lambda < 0$; $R(0) = 0 \Uparrow$; BB; Harrison F,1; Bondi 1iii

We have already investigated a model where $\lambda < 0$ in model#7. In that model, $k = 0$. Now we are considering what happens when both the lambda force and the curvature term are tending to contract the universe. R is a decreasing function, positive when $0 < R < R_c$, and negative for $R > R_c$, where R_c is the critical scaling factor where expansion turns into contraction. The expansion begins as in the Einstein-de Sitter model (model#1), equ(4.7)(17). The addition of $k < 0$ to the model simply slows the expansion sooner and causes the contraction to begin earlier.

This model is similar to models #3, #7, #14, #22 and #23.

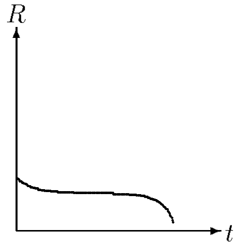


MODEL# 18

Eddington-Lemâitre

$k > 0$; $\lambda = \lambda_E$; $R(0) > 0 \Rightarrow$; SB; Harrison 7; Bondi -

This model is the time-reversal of model#10.

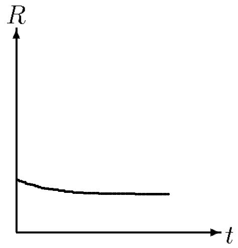


MODEL# 19

Lemâitre

$k > 0$; $\lambda = \lambda_E$; $R(0) > 0 \Downarrow$; WSB; Harrison 13; Bondi -

This model is the time-reversal of model#13.

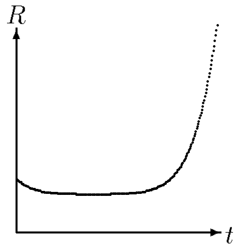


MODEL# 20

Eddington-Lemâitre

$k > 0$; $\lambda = \lambda_E$; $R(0) > 0 \Downarrow$; WS; Harrison 11; Bondi -

This model is the time-reversal of model#11.

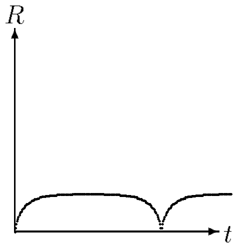


MODEL# 21

Bounce

$k > 0$; $\lambda = \lambda_E$; $R(0) > 0 \Downarrow$; WSW; Harrison 14; Bondi 3i

This model is a combination of model#20 and its time-reversal, model#11.



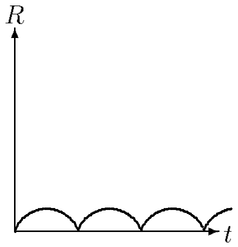
MODEL# 22

Closed with static phase

$k > 0$; $\lambda = \lambda_E$; $R(0) = 0 \Uparrow$; BSB; Harrison 4; Bondi -

This model is the same as model#3, except the lambda term forces a brief static phase.

It is similar to models #3, #7, #14, #17 and #23.



MODEL# 23

Closed quickly

$k > 0$; $\lambda < 0$; $R(0) = 0 \Uparrow$; BB; Harrison K; Bondi -

This model is the same as model#3, except the lambda term is negative, increasing the rate at which the scaling factor changes. The arguments given in model#7 and model#17 apply here, since $\lambda < 0$.

This model is similar to models #3, #7, #14, #17 and #22.

Chapter 6

Non-Friedmann Models

Many models of the universe are not based on the Friedmann Equation. This chapter briefly presents a few of those models. At the end of each model's description is a list of a few references which discuss the model at greater length. They are not necessarily the best references... only the ones I have come across.

Steady State: Perhaps the most famous non-standard model is the Steady State model proposed by Bondi, Hoyle and Gold in 1948. The model extends the normal list of assumptions made about the universe to include the "Perfect Cosmological Assumption", which states the large scale properties of the universe are independent of location (homogenous), direction (isotropic), and time. The universe of the Steady State model is infinitely old and infinitely young, but is nevertheless expanding. This is only possible if there is continuous creation of matter to maintain a constant density. Hoyle added a C-field (short for Creation-Field) to the field equations of general relativity to force the creation of this matter. Because nothing changes in the Steady State universe, the curvature k, the Hubble term H, and the deceleration term q must all remain constant. Matter need only be created at the rate of 1 hydrogen atom per cubic meter every 5 billion years. The Steady State universe completely regenerates itself in $1/(3H)$. The strongest refutation to the Steady State model is the background radiation, which is easily explained as the ashes of the Big Bang. Supporters of the Steady State theory look to astrophysical processes to explain the background radiation. For more details of the Steady State universe, I refer you to Bondi (1952) p140-156, Harrison (1981) p318-320, Harrison (1981) p295-296, Harrison (1981) p313-314, Narlikar (1977) p131-137, Rowan-Robinson (1985) p249, Silk (1980) p318-319, Hoyle (1975) p675-681, Terzian (1982) p1-60, Peebles (1993) §7.

Tired-Light: The expansion of the universe inferred by the redshift of distant galaxies has proven to be the most important contribution of the 20th century to cosmology. Yet the nagging thought lingers: could we be completely wrong in our interpretation of the redshift? No one has ever actually proven that the redshift is caused by the expansion of the space-time. In principle, light could be redshifted by many other effects. One such effect could be "tired light"; a quantum of light could lose energy during its journey through space from remote objects. It certainly can be argued that, over the vastness of intergalactic space, our terrestrial laws of physics may be wholly inapplicable or at least incomplete. Silk (1980) p316-317, Harrison (1981) p240-241, Unknown (1986) p64, Peebles (1993) §7.

Variable-Mass: Another effect that could cause a redshift would be if the masses of particles in the distant (therefore older) regions of space were less than the masses that we measure on Earth. If this were so, less energy would be required to boost an electron into a higher energy level, and less would be released when the electron fell back down to a lower energy level and released radiation that we eventually see. This would appear to us as a redshift. The most complete theory I know of that proposes such an effect is the Hoyle-Narlikar theory of Conformal Gravity. This theorizes that particles are "born" massless and acquire mass through their interaction with the rest of the particles in the universe. Since a particle's horizon expands, it gains mass. Hoyle states that a cosmology based on constant particle masses and complex universal geometry (eg, the Big Bang with its expansion) is completely equivalent to a cosmology where the particle masses vary and the universal geometry is simple. The theory is an application of Mach's Principle. The variable mass theories are closely related to the variable-G theories, explained next. Harrison (1981) p322-323, Narlikar (1977) p137-139, Rowan-Robinson (1985) p170-175, Rowan-Robinson (1985) p250, Hoyle

(1975) p657-669, Arp (1987) p178-184.

Variable G: In our analysis of the models so far, we have assumed G to be a constant that does not change with location or time. In instead we allow G to vary, we have a variable-G cosmology. Theories of this type are related to the large-number hypothesis which states that the coincidences in the large dimensionless constants of nature (10^{40}, 10^{80}) are not coincidences at all but are true constants that remain the same because all of their constituents (such as G) change in time in ways that keep the dimensionless numbers constant. Another variation on this type of theory presumes that rather than the universe expanding, matter is really shrinking. Harrison (1981) p322-323, Narlikar (1977) p137-139, Narlikar (1977) p170-175, Rowan-Robinson (1985) p250, Rowan-Robinson (1985) p321, Hoyle (1975) p657-669.

Cold Big Bang: If the matter in the Big Bang was cold rather than hot, galaxy formation would have been easier at earlier times and several problems with the Big Bang theory could be solved. Alas, the background radiation would be unexplained, so proponents of cold Big Bang models support the idea that the background radiation is made by astrophysical processes. Rowan-Robinson (1985) p250, Silk (1980) p326.

Inflationary universe: A period of super-rapid expansion happens during the early Big Bang in this model. This super-fast expansion is caused by the universe being born in other than its lowest-energy state (a so-called false vacuum). This inflation may force any initial conditions of the universe into a flat (not open nor closed) and homogenous universe, thus explaining two outstanding cosmological problems with the standard Big Bang model. This model is becoming more and more widely accepted. Silk (1980) p250, Guth (1984).

Alfven-Klein Cosmology: This model answers the question of why we live in a matter-dominated universe by postulating that we don't. A primordial mix of matter and antimatter causes intense radiation that forces expansion which separates matter regions from antimatter regions, which are still in the universe, according to this theory. Silk (1980) p319-320.

Chaotic Cosmology: The problem of galaxy formation is solved by assuming a lumpy early universe which has been smoothed by effects that take place after the Big Bang. Rowan-Robinson (1985) p250, Harrison (1981) p315-316.

Electromagnetic-Dominated universe (plasma universe): This model, supported mainly by Alfven and Lerner, suggests that plasma gathered near huge magnetic and electrical fields in the universe, and electromagnetism therefore dominates the large-scale structure of the universe rather than gravity. This model is suggestively supported by huge filamentary structures now being seen in large-scale maps of galaxies, and recent observations of magnetic fields in clusters of galaxies. More information can be found in *The Big Bang Never Happened*, by Eric Lerner, and Lerner (1988a) p70-79, Peratt (1985) p389, Lerner (1988b) p118, Kanipe (1992) p32-37, Peratt (1992) p136-140, Lerner (1992) p124, Peebles (1993) §7.

Mach's Principle: This principle states that the motion of anything is due to the effects of the rest of the universe. One is the Hoyle-Narlikar theory mentioned above, and another is the Spinning universe theory, described below. Almost any astronomy book will mention Mach's Principle. Perhaps one of the better sources is Harrison (1981) p176-179.

Spinning universe: Proposed by Godel and Oszvath. I do not understand this theory, and refer you only to Narlikar (1977) p169.

Scalar-Tensor Theory: G, c, and/or M can vary from place to place in the universe. Harrison (1981) p320-324.

Fractal Cosmology: This model assumes that the distribution of galaxies in the universe is not homogenous, but rather is fractal. Peebles (1993) §7.

Chapter 7

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